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Localization for Random and Quasiperiodic Potentials

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A survey is made of some recent mathematical results and techniques for Schrödinger operators with random and quasiperiodic potentials. A new proof of localization for random potentials, established in collaboration with H. von Dreifus, is sketched.

KEY WORDS: Localization; random potentials; Schrödinger operator; percolation; quasiperiodic.

1. INTRODUCTION

The main purpose of this note is to sketch a new proof of localization for the random Schrödinger operator

$$H = -\Delta + \lambda v \tag{1}$$

acting on $l_2(\mathbb{Z}^d)$. Here Δ denotes the finite-difference Laplacian and $v(j)$, $j \in \mathbb{Z}^d$, are assumed to be independent random variables with a common bounded distribution density $g(v_j)$. We establish localization (pure point spectrum) when either λ is large or when λ is small and the energies we are considering lie in the band tail. Later we describe some recent results obtained for the case where v is quasiperiodic.

When v is random the spectrum of H is known as a set

$$\operatorname{spec} H = \operatorname{spec} - \Delta + \lambda \operatorname{supp} g$$

with probability one. If, for example, g is the characteristic function of $[-1/2, 1/2]$, then

$$\operatorname{spec} H = [0, 4d] + \lambda[-1/2, 1/2] = [-\lambda/2, 4d + \lambda/2]$$

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The qualitative nature of the spectrum of H is a more difficult problem. Of course, if $\lambda = 0$ or v is periodic, the spectrum of H is absolutely continuous and its generalized eigenfunctions are Bloch waves $p(j)\, e^{ik \cdot j}$ where p is a periodic function. These states are said to be extended. However, if $d = 1$ and $v(j)$ is random as above, then H has pure point (p.p.) spectrum with probability one for all $\lambda \neq 0$. Its eigenstates decay exponentially fast about some point in space and so are said to be *localized*. This was first proven in the continuum by Goldsheid *et al.*[1] The same results hold in any dimension for large disorder, i.e., $\lambda \gg 1$, as was first predicted by Anderson.[2]

When $d \geqslant 3$ and λ is small one expects there to be a band of absolutely continuous (a.c.) spectrum. Nevertheless, the existence of a.c. spectrum remains an open and apparently difficult mathematical problem. For any $\lambda \neq 0$, near the end points of the spectrum, i.e., the band tails, it is known that there is always an interval of spectrum that is pure point. See Fig. 1.

In two dimensions it is conjectured that all states are localized for any $\lambda \neq 0$. See ref. 3 for a more detailed mathematical survey and further references.

The main estimate needed to establish localization can be expressed in terms of the decay of the Green's function

$$G(E + i\varepsilon, x, y) = [H - E - i\varepsilon]^{-1}(x, y)$$

where $x, y \in \mathbb{Z}^d$. We shall frequently need a finite-volume Green's function. For $\Lambda \subset \mathbb{Z}^d$, let $H(\Lambda)$ denote H restricted to Λ with Dirichlet boundary conditions on the boundary of Λ, $\partial\Lambda$. We write the corresponding Green's function as G_Λ. A finite-volume criterion for localization can now be formulated as follows: Let Λ_l be a cube of side $2l$ with center c. We define

$$P_l(E) = \text{prob}\left\{ \max_{|a-c| \leqslant l/2} \sum_{b \in \partial\Lambda_l} |G_{\Lambda_l}(E, a, b)|\; l^2 \geqslant \tfrac{1}{2} \right\} \tag{2}$$

$P_l(E)$ may be interpreted as the probability that a point inside Λ_l feels the boundary. See Fig. 2. Given a particular v, if the inequality (2) holds, then Λ_l is said to be *l-singular*; otherwise, it is said to be *l-regular*. Hence $P_l(E)$ is the probability that Λ_l is l-singular.

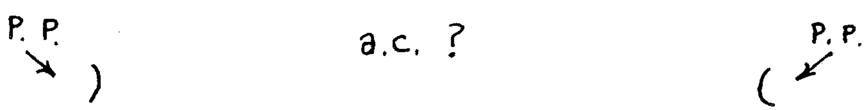

Fig. 1. The spectrum of H for small λ and $d \geqslant 3$.

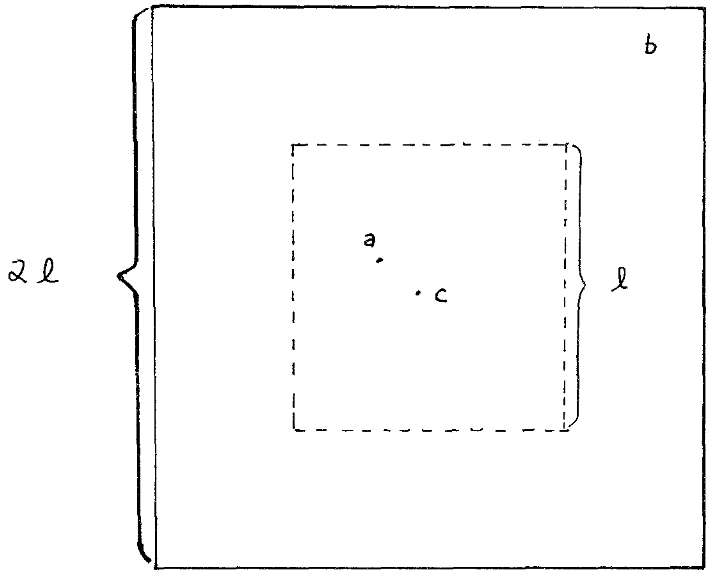

Fig. 2. The cube Λ_l of side $2l$ with center c.

Theorem 1. There is a $\delta_0 > 0$ such that if $P_l(E) \leqslant \delta_0$ for some $l \geqslant 4$, then for all E' such that $|E - E'| \leqslant \delta_1(l) > 0$,

$$\sup_{\varepsilon \neq 0} |G(E' + i\varepsilon; x, y)| \leqslant c_x e^{-\gamma |x-y|} \tag{3}$$

For each fixed E', the constant c_x is finite with probability one and const $\gamma \cong 1/l$.

By a theorem of Simon and Wolff[4] and of Delyon *et al.*[5] if H is random (in a weak sense) and its Green's function satisfies (3), then H has pure point spectrum in the interval $|E' - E| \leqslant \delta_1(l)$ and the eigenstates decay exponentially fast. The constant $\delta_1(l)$ is chosen small, $\simeq l^{-(2d+2)}$, so that $P_l(E') \sim P_l(E)$ for E' in the interval.

In one dimension $G(E)$ decays exponentially fast with probability one for any fixed E and $\lambda \neq 0$. No hypothesis on $P_l(E)$ is needed. This follows from Furstenberg's theorem[6] on products of $Sl(2)$ random matrices, which tells us that with probability one for any fixed E, $H\psi = E\psi$ has an exponentially growing solution, i.e., $\psi(n) \sim e^{\gamma |n|}$ as $n \to \infty$ or $n \to -\infty$. The Green's function can be formed from these solutions.

In any dimension, if λ is large and we set $l = 4$, it is not hard to check that $P_l(E)$ is small. This is because $|\lambda v(j) - E|$ is large with high probability. From this we see that the Green's function in Λ_l is small, so P_l is small. This proves localization for $|\lambda| \gg 1$. When E belongs to the band tail, $P_l(E)$ is small for large l. The argument here is more complicated: Let $A = \inf \operatorname{spec} H$. It is known that the probability that $H(\Lambda_l)$ has an eigenvalue in the interval $[A, A+2\delta]$ is less than const $\times\, l^d \exp(-c_1 \delta^{-d/2})$ for some constant $c_1 > 0$.[7,8] This is Lifshitz's result on density of states in the band tail. On the other hand, if $\operatorname{spec} H(\Lambda) \cap [A, A+2\delta] = \varnothing$ and E is in the band tail $[A, A+\delta]$, then

$$|G_\Lambda(E, a, b)| \leqslant (1/\delta) \exp(-\delta l/\text{const})$$

provided $|a-b| \geqslant l/2$. This is a standard fact for Schrödinger operators, which only uses the fact that $\operatorname{dist}(E, \operatorname{spec} H(\Lambda)) \geqslant \delta$.[3,9] If we choose $l = \delta^{-3/2}$ and δ small, we see that $|G_\Lambda(E, a, b)|$ is small with high probability, so $P_l(E)$ is small.

To obtain (3) for $d \geqslant 2$, Fröhlich and Spencer[10,3] developed a multiscale perturbation scheme analogous to a probabilistic KAM approach, which made it possible to control small denominators that appear in perturbation theory. Small denominators naturally appear in estimates on $G(E)$ in the form $(E^i - E)^{-1}$, where E^i are eigenvalues of H that come arbitrarily close to E. In refs. 10 and 11 we kept track of the "position" and "strength" of the small divisors. This technique is also particularly useful in the analysis of quasiperiodic potentials explained later. The new proof I sketch here for the random case is due to Henrique von Dreifus[12] and myself. We use many ideas of ref. 10, but this new approach is technically simpler, particularly the probabilistic estimates. We were motivated by scaling ideas used in bond percolation, which were explained to us by Jeniffer and Lincoln Chayes. See ref. 13. In percolation $P_l(E)$ may be thought of as the probability that opposite sides of a cube of side l are connected by occupied bonds. Localization in this language is then just the absence of percolation.

2. SKETCH OF THE PROOF OF THEOREM 1

To illustrate the proof of Theorem 1, we shall prove a weaker form of it, namely if $P_{l_0}(E) \leqslant \delta_0$, then $P_l(E) \to 0$ as $l \to \infty$. From this result it can be shown that $|G(E, 0, x)|$ is summable in x with probability one. By refs. 4 and 5, H has pure point spectrum near E with summable eigenfunctions. The exponential decay of Theorem 1 can be obtained by modifying our definition of P_l for $l \gg l_0$. We introduce an additional exponential factor in P_l and use a sequence of length scales $l_i = \exp[\beta(3/2)^i]$. See ref. 12 for details.

Now assume P_l is small and let $L = Rl$. The key estimate is now

$$P_L(E) \leqslant CP_l(E)^2 R^{2d} + 2M(2L)^d l^{-R/4} \tag{4}$$

where $M = \max g(v)$. It is easy to see that for large R independent of l, e.g., $R = 10d$, $P_{l_i} \to 0$, where $l_i = R^i l$.

The proof of (4) begins by dividing Λ_L into a lattice of subcubes of side $2l$ as in Fig. 3. There are R^d such cubes. In addition, we consider a larger family $\mathscr{F}_l(\Lambda_L)$ of overlapping cubes of side $2l$. This family includes the original family and shifted lattices of cubes so that intersections of the type indicated in Fig. 4 are allowed. The family $\mathscr{F}_l(\Lambda_L)$ has the property that for any point x in Λ_L there is a cube $\Lambda_l \in \mathscr{F}_l$ such that $x \in \Lambda_l$ and $|x - c| \leqslant l/2$, where c is the center of Λ_l. Moreover $\mathscr{F}_l(\Lambda_L)$ has at most $C_d R^d$ elements.

The proof of (4) may now be divided into three parts:

I. If all cubes of $\mathscr{F}_l(\Lambda_L)$ are l-regular, then Λ_L is L-regular. This assertion will follow by expressing G_{Λ_L} as a series in G_{Λ_l} and by using

$$\sum_{b \in \partial \Lambda_l} |G_{\Lambda_l}(E; a, b)| \leqslant \frac{1}{2l^2} \tag{5}$$

for all $|a - c| \leqslant l/2$. This inequality is simply the statement that $\Lambda_l \in \mathscr{F}_l$ is l-regular.

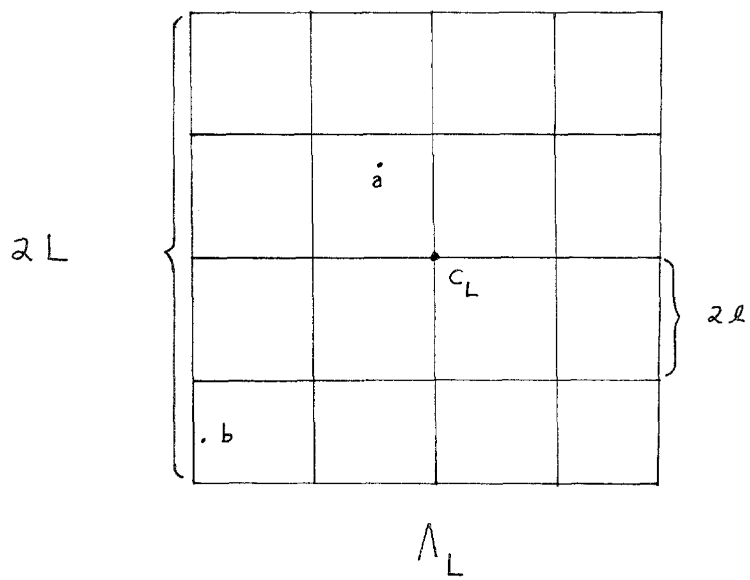

Fig. 3. Family of overlapping cubes Λ_L, $|a - c_L| \leqslant L/2$.

Fig. 4. Overlaps permitted in $\mathscr{F}_l(\Lambda_L)$.

II. The probability that there are two or more disjoint cubes in $\mathscr{F}_l$ that are l-singular is less than

$$(C_d^2 R^{2d})\, P_l^2(E)$$

The first factor is an entropy factor, which counts the number of possible pairs of cubes. The second factor follows from the independence of the disjoint l-singular cubes. This product gives the first term in our upper bound on P_L.

III. If there is only one l-singular cube and if

$$\|G_{\Lambda_L}(E)\| \leqslant l^{R/4} \tag{6}$$

then Λ_L is L-regular. The same statement holds if there are several l-singular cubes, no two of which are disjoint.

There is a theorem due to Wegner[14,3] which implies that the probability that (6) fails is less than $2M(2L)^d\, l^{-R/4}$. This accounts for the final term of (4). It is the basic estimate that controls the probability that a small denominator $[E^i(\Lambda_L) - E]^{-1}$ appears.

Now we provide some further details for I and III. Assertion II is clear. We shall use the resolvent identity in the following form. Let $A \subset \Lambda$; then

$$H(\Lambda) = H(A) + H(\Lambda \backslash A) - \Gamma$$

where Γ is the operator coupling A to $\Lambda \backslash A$ through the boundary of A in Λ. The operator Γ has matrix elements

$$\begin{aligned} \Gamma_{ij} &= 1 \qquad (i, j) \in \partial A, \quad (i, j) \notin \partial \Lambda \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Here the boundary of A is the set of unordered pairs (i, j) such that $i \in A$,

$j \notin A$. For notational simplicity we usually write $\Gamma = \partial A$. For $x \in A$, the desired resolvent identity is

$$G_A(x, y) = G_A(x, y) + \sum_{z, z'} G_A(x, z)\, \Gamma_{z, z'} G_A(z', y) \tag{7}$$

As above, we now suppress the dependence of G on E.

To prove I, we iterate (7) with $A = A_L$ and for some sequence $A_i \in \mathscr{F}_l(A_L)$ centered at c_i. To begin, let a and b be as in Fig. 3, and $A_0 \in \mathscr{F}_l$ such that $a \in A_0$ and $|a - c_0| \leqslant l/2$. Set $\Gamma^i = \partial A_i$, $G_i = G_{A_i}$. Now, since $G_0(a, b) = 0$, we have

$$G_A(a, b) = \sum_{z_1, z_1'} G_0(a, z_1)\, \Gamma^0_{z_1 z_1'} G_A(z_1', b) \tag{8}$$

To iterate (8), we pick a cube $A_1 = A_1(z_1')$ containing z_1' and satisfying $|c_1 - z_1'| \leqslant l/2$. Now G_A becomes

$$G_A(a, b) = \sum_{z} G_0(a, z_1)\, \Gamma^0_{z_1 z_1'} G_1(z_1', z_2)\, \Gamma^1_{z_2 z_2'} G_A(z_2', b) \tag{9}$$

We continue to iterate and obtain the *block resolvent expansion.* A chain A_i is sketched in Fig. 5. Since the elements of $\mathscr{F}_l$ are all l-regular, each sum over z_i by (5) gives a factor of l^{-2}. Thus, we have

$$|G_A(a, b)| \leqslant (l^{-2}/2)^{R/4} \leqslant L^{-(d+2)}$$

for large R; hence I holds. The exponent $R/4$ comes from the fact that there are at least $R/4$ factors of G_i before we reach the boundary. In other words, for some $i \geqslant R/4$, $G_i(z_i', b) \neq 0$. See Fig. 5. Some details concerning the expansion as we get near the ∂A are omitted. See ref. 12.

To prove III, let $\bigcup$ denote the union of the *intersecting* l-singular cubes. Clearly $\bigcup$ is contained in a cube of side $4l$. Note that each cube of $\mathscr{F}_l$ in $A \equiv A_L \backslash \bigcup$ is l-regular, and from (5) and the block resolvent expansion,

$$|G_A(x, y)| \leqslant l^{-2r} \tag{9}$$

where $r = |x - y|/2l \geqslant 2$. We iterate (7) once with $A = A_L$ and $A = A_L \backslash \bigcup$ and obtain

$$\begin{aligned} G_A(a, b) &= [G_A + G_A \Gamma G_A](a, b) \\ &= [G_A + G_A \Gamma G_A + G_A \Gamma G_A \Gamma G_A](a, b) \end{aligned} \tag{10}$$

where $\Gamma = \partial \bigcup$. First we estimate the third term on the right side of (10). In

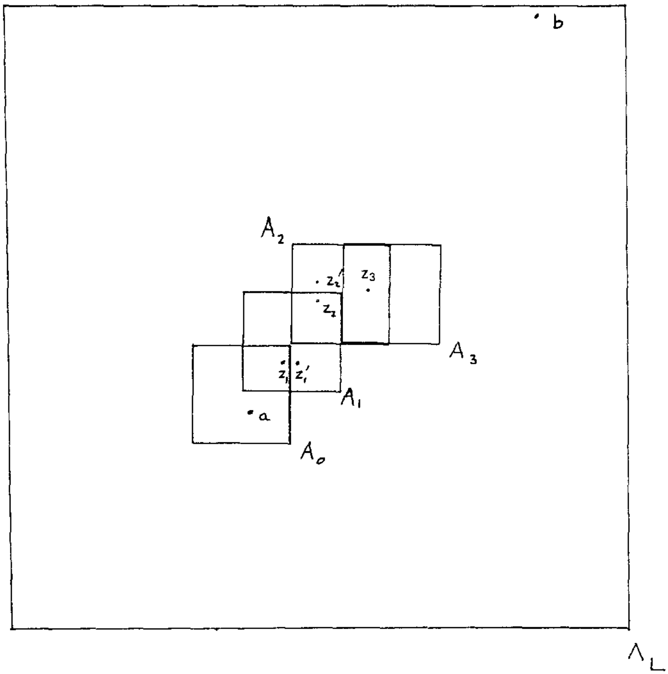

Fig. 5. A chain of cubes in $\mathscr{F}_l$ appearing in the block resolvent expansion.

matrix form the third term can be expressed as a sum over z_1, z_1' and z_2, z_2' over $\partial \bigcup = \Gamma$. We claim that this sum is bounded by

$$\sum_z |G_A(a, z_1)\, G_A(z_1', z_2)\, G_A(z_2', b)|$$

$$\leqslant (2l)^{2(d-1)} \frac{1}{l^{2r_1}}\, l^{R/4} \frac{1}{l^{2r_2}} \leqslant 2L^{-(d+2)} \qquad \text{for large } R$$

The first factor on the right accounts for the sum over $\partial \bigcup$. We have used (9) with $r_1 = |a - z_1|/2l$, $r_2 = |z_2 - b|/2l$, and $r_1 + r_2 \geqslant R/4$ and (6) to bound $G_A(z_1, z_2)$. The first and second terms can be estimated in a similar fashion. Hence, Λ_L is L-regular. This completes our sketch.

3. QUASIPERIODIC POTENTIALS

We now review some recent work on quasiperiodic potentials. We shall consider two Hamiltonians:

$$H = -\varepsilon^2 \Delta + \cos 2\pi(j\alpha + \theta) \tag{11}$$

on $l_2(\mathbb{Z}^1)$ and

$$H_c = -\frac{d^2}{dx^2} + K^2[\cos x + \cos(\alpha x + \theta)] \tag{12}$$

on $L_2(R)$, where α is an irrational satisfying the Diophantine condition,

$$\text{const} \times |j|^2 \, |\sin(j\alpha 2\pi)| \geqslant 1$$

The first mathematical analysis of H_c is due to Dinaburg and Sinai,[15] who proved the existence of some absolutely continuous spectrum at high energy for any value of K. The corresponding eigenstates are of Bloch type $qp(x)\, e^{ik \cdot x}$, where qp is quasiperiodic. These results were obtained by KAM techniques to overcome small denominator singularities. The coexistence of a point spectrum for H_c remained an open question.

On the lattice, using the self-duality of H, it was shown that for any irrational α, if $\varepsilon^2 < 1/2$, H has no absolutely continuous spectrum.[16–18] On the other hand, if $\varepsilon^2 > 1/2$, H has no point spectrum.[16,19] In ref. 20 the existence of a point spectrum was established for small ε provided α satisfied a Diophantine condition. When α is irrational but well approximated by rationals and $\varepsilon^2 < 1/2$, the spectrum of H is purely singular continuous.[18]

Many of the techniques and results described above do not extend to the continuum and in fact are special to the cosine potential. Recently, Fröhlich *et al.*,[21] using some of the techniques of refs. 10 and 11, established a pure point spectrum for H_c at low energy provided K is large. We prove that the eigenfunctions decay exponentially fast and have precisely 2^n "peaks" for some $n = 0, 1, 2, \ldots$. The spectrum of H_c is essential, which means that there are no isolated eigenvalues. On the lattice, if ε is small and the cosine in (11) is replaced by an even, C^2 periodic function with exactly two nondegenerate critical points, we prove that H has pure point spectrum. This result was independently proved by Sinai,[22] who in addition showed that the integrated density of states is an incomplete devil's staircase.

Returning to the continuum, when K in (12) is small, Surace[23] has proved that H_c has no eigenvalues and we conjecture that H_c has purely absolutely continuous spectrum. When K is large it follows from ref. 23 that H has no eigenstates at high energy. The nature of the spectrum of H_c at intermediate energies remains unclear. Is there a single mobility edge E_m below which there is only point spectrum and above which there is no point spectrum? Or is there a band $[E_m^-, E_m^+]$ of mobility edges where the spectrum is a complicated mixture of pure point and absolutely continuous states?

We now sketch a proof of the absence of point spectrum for H_c when K is small. The strategy is due to Aubry[16] and for mathematical details see refs. 19 and 23. Suppose $f(x)$ is a square integrable eigenfunction of H_c for some θ with eigenvalue E. Let $\hat{f}(p) \in L_2(R)$ denote the Fourier transform of f and define

$$a_{m,n}(p) = \hat{f}(p + m + n\alpha)\, e^{+in\theta}$$

where $(m, n) \in \mathbb{Z}^2$. It is straightforward to show that $a_{mn}(p)$ is well defined for almost all p and $|a_{mn}(p)| \leqslant O(m^2 + n^2)^2$. Furthermore, $a(p)$ is an eigenfunction of the dual operator $\hat{H}(p)$:

$$\begin{aligned}\hat{H}_c a &\equiv \tfrac{1}{2}K^2(a_{m+1,n} + a_{m-1,n} + a_{m,n+1} + a_{m,n-1}) + (m + n\alpha + p)^2\, a_{m,n} \\ &= (K^2\Delta + v)a = Ea \qquad (13)\end{aligned}$$

where $v(p) = v = (m + n\alpha + p)^2$ and Δ is our finite-difference Laplacian on $\mathbb{Z}^2$ with diagonal matrix elements set equal to zero. What Surace proves is that $\hat{H}_c$ and H share similar qualitative features when K and ε are small. In particular, using techniques of ref. 21, he proves that $\hat{H}_c$ has pure point spectrum for almost all $p \in R$ with eigenvalues that depend "sensitively" on p. Now to show f is zero, i.e., there are no eigenfunctions for H_c, it suffices to prove that $a(p) = 0$ for almost all p. Roughly speaking, $a(p) = 0$, because the eigenvalue E of $a(p)$ is independent of p. More precisely, let Λ_i be a sequence of squares in $\mathbb{Z}^2$ of side $l_i \to \infty$ and let $\hat{G}_i(E)$ denote the Green's function of $\hat{H}(\Lambda_i)$. It follows from the results of ref. 23 that for E independent of p, if $|x - y| \geqslant l_i/4$ and $i \geqslant i_0(p)$, then

$$|\hat{G}_i(E, x, y)| \leqslant e^{-|\gamma| \cdot |x-y|} \qquad x, y \in \mathbb{Z}^2 \qquad (14)$$

with $i_0(p) < \infty$ for almost all p. Using $\hat{G}_i$, we can recover a from its values on the boundary Γ^i of Λ_i:

$$a_x = \sum_{y, y'} \hat{G}_i(E, x, y)\, \Gamma^i_{yy'} a_y$$

Hence, from the exponential decay of $\hat{G}$ and the polynomial growth of a_y

$$|a_x| \leqslant e^{-l_i/2} O(l_i^2 \cdot l_i^4)$$

Finally, since $l_i \to \infty$, we conclude $a_x = 0$.

Note that in the random case the analogue of (14) is (3). The main problem in the quasiperiodic problem is that of course there is no independence and the only notion of randomness is in the parameter θ or p. All statements made in this section hold with "probability one," i.e., for almost all θ or p with respect to Lebesgue measure.

REFERENCES

1. I. Goldsheid, S. Molchanov, and L. Pastur, *Funct. Anal. Appl.* **11**:1 (1977).
2. P. Anderson, *Phys. Rev.* **109**:1492 (1958).
3. T. Spencer, in *Critical Phenomena, Random Systems and Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, 1986).
4. B. Simon and T. Wolff, *Commun. Pure Appl. Math.* **39**:75 (1986).
5. F. Delyon, Y. Levy, and B. Souillard, *Commun. Math. Phys.* **100**:21 (1985).
6. H. Furstenberg, *Trans. Am. Math. Soc.* **108**:377 (1963).
7. I. M. Lifshitz, *Sov. Phys. Usp.* **7**:549 (1965).
8. W. Kirsch and F. Martinelli, *Commun. Math. Phys.* **89**:27 (1983).
9. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV* (Academic Press, New York, 1978).
10. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **88**:151 (1983).
11. J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, *Commun. Math. Phys.* **101**:21 (1985).
12. H. von Dreifus, NYU. Thesis, Part 2.
13. J. Chayes and L. Chayes, in *Critical Phenomena, Random Systems and Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, 1986).
14. F. Wegner, *Z. Phys. B* **44**:9 (1981).
15. E. Dinaburg and Ya. Sinai, *Funct. Anal. Appl.* **9**:279 (1975).
16. S. Aubry, *Solid State Sci.* **8**:264 (1978).
17. M. Herman, *Comm. Math. Helv.* **58**:453 (1983).
18. J. Avron and B. Simon, *Duke J.* **50**:369 (1983).
19. F. Delyon, *J. Phys. A* **20**:L21 (1987).
20. J. Bellissard, R. Lima, and D. Testard, *Commun. Math. Phys.* **88**:207 (1983).
21. J. Fröhlich, T. Spencer, and P. Wittwer, preprint.
22. Ya. Sinai, *J. Stat. Phys.* **46** (1987).
23. S. Surace, NYU. Thesis.